

# ON THE $\zeta_3$ PELL EQUATION

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ABSTRACT. Let  $K = \mathbb{Q}(\zeta_3)$ , where  $\zeta_3$  is a primitive root of unity. In this paper we study the distribution of integers  $\alpha \in \mathcal{O}_K$  for which the norm equation  $N_{K(\sqrt[3]{\alpha})/K}(\mathbf{x}) = \zeta_3$  is solvable for integers  $\mathbf{x} \in \mathcal{O}_{K(\sqrt[3]{\alpha})}$ . The analogous question for  $\zeta_2 = -1$  is the well-known negative Pell equation.

## 1. INTRODUCTION

A classical question in number theory is whether there is a solution to the Pell equation  $x^2 - Dy^2 = 1$  in integers  $x$  and  $y$  for  $D > 1$  a square-free integer. While this has an affirmative answer for all  $D$ , the standard proof of this fact actually first shows that there is a solution to  $x^2 - Dy^2 = \pm 1$  and then uses that to produce a solution to  $x^2 - Dy^2 = 1$ . Thus, one is led to ask questions about the negative Pell equation  $x^2 - Dy^2 = -1$ . Many partial results have been obtained (e.g. [5] and [1]). The question about the negative Pell equation comes down to one about whether there is a unit in  $\mathbb{Q}(\sqrt{D})$  which has norm  $-1$ . Viewed from this perspective, it becomes possible to generalize this question to other classes of cyclic extensions. In this paper, we will be focusing on the next simplest case.

Let  $K = \mathbb{Q}(\zeta_3)$ . Then, by Kummer theory, all cyclic cubic extensions of  $K$  are of the form  $K(\sqrt[3]{\alpha})$  for  $\alpha \in K^*$ . Motivated by the previous discussion, we are interested in whether there is a unit in  $K(\sqrt[3]{\alpha})$  with norm  $\zeta_3$ . One appealing thing about this problem is that some of the structure in the negative Pell equation problem goes away, but there are still many similarities. If one replaced 3 with a larger prime, issues about global units would come up, and if the choice of prime was irregular, issues about relative class groups would arise.

Readers familiar with the negative Pell equation know that there are local obstructions that need to be satisfied for there to be a chance of a solution to the equation; in particular, every odd prime dividing  $D$  must be congruent to 1 mod 4. A similar issue arises here: one needs that every prime  $\pi$  dividing  $\alpha$  to have  $N_{K/\mathbb{Q}}(\pi) \equiv 1 \pmod{9}$  or have  $\pi$  being an associate of  $1 - \zeta_3$ , the prime above 3. We will restrict ourselves to the set of all  $\alpha$  such that  $K(\sqrt[3]{\alpha})/K$  is unramified at  $1 - \zeta_3$ , as the case where the ramification at  $1 - \zeta_3$  is like  $K_{1-\zeta_3}(\zeta_9)/K_{1-\zeta_3}$  provides no local obstruction to there being a solution to the  $\zeta_3$ -Pell equation but behaves very differently from the other cases. Put

$$(1.1) \quad S(X) = \{\alpha \mid N_{K/\mathbb{Q}}(D_{K(\sqrt[3]{\alpha})/K}) < X, N_{K/\mathbb{Q}}(\pi) \equiv 1 \pmod{9} \forall \pi \mid \alpha\}$$

and let

$$(1.2) \quad S^{\zeta_3}(X) = \{\alpha \in S(X) \mid \exists u \in K(\sqrt[3]{\alpha}) \mid N_{K(\sqrt[3]{\alpha})/K}(u) = \zeta_3\}.$$

We are interested in the size  $\frac{|S^{\zeta_3}(X)|}{|S(X)|}$  as  $X \rightarrow \infty$ . To state the main theorem of the paper, let

$$(1.3) \quad \beta = \prod_{i \geq 0} \left(1 - \frac{1}{3^{2i+1}}\right) = \prod_{j=1}^{\infty} (1 + 3^{-j})^{-1}.$$

This constant is the limit of the probability that a random large symmetric matrix over  $\mathbb{F}_3$  has full rank. Our main theorem is then the following:

**Theorem 1.1.** *One has that*

$$\beta \leq \liminf_{X \rightarrow \infty} \frac{|S^{\zeta_3}(X)|}{|S(X)|} \leq \limsup_{X \rightarrow \infty} \frac{|S^{\zeta_3}(X)|}{|S(X)|} \leq \frac{3}{4},$$

where  $\beta$  is given as in (1.3).

The proof of this theorem is very simple if one believes some analytic estimates. The proof is in Section 5. Following Stevenhagen, one might be tempted to make the following conjecture:

**Conjecture 1.2.** *One has that*

$$\lim_{X \rightarrow \infty} \frac{|S^{\zeta_3}(X)|}{|S(X)|} = 2 - 2\beta.$$

We believe this conjecture is false, even when one assumes that there is no ramification at  $1 - \zeta_3$ . We have no theoretical explanation for why this is but there are numerical calculations that cast doubt on the underlying heuristic that produces this number.

This paper is in three parts. The first part will be algebraic criteria for having a solution or no solution to the  $\zeta_3$ -Pell equation. The second part will be analysis to patch together the algebraic criteria to get analytic results. The third part is then a discussion of why conjectures are different from the negative Pell case.

Finally, the astute reader may notice that this theorem is very similar to the one in [4]. Indeed, the overarching strategy is the same, and the lower bound is argued in a similar manner. The upper bound is made complicated by the lack of an analogue of the narrow class group, and so we have to use techniques that were developed in [6] to get around this.

## 2. GENUS THEORY AND RÉDEI MATRICIES

This section will discuss genus theory and especially how it applies to the case of extensions of the form  $K(\sqrt[3]{\alpha})/K$ . The main goal is to come up with algebraic criteria that characterize whether there is a unit  $u \in \mathcal{O}_{K(\sqrt[3]{\alpha})}$  with norm equal to  $\zeta_3$ .

**2.1. The Cubic Residue Symbol and Cubic Reciprocity.** Let  $\mathbb{Z}[\zeta_3]$  be the ring of Eisenstein integers. This ring is a principal ideal domain and it has the units  $\pm 1, \pm \zeta_3, \pm \zeta_6$ . Up to units, the irreducible elements are  $\lambda = 1 + \zeta_3$ , the rational primes  $q \equiv 2 \pmod{3}$ , and the elements  $\pi$  of  $\mathbb{Z}[\zeta_3]$ , such that  $\pi\bar{\pi}$  is a rational prime  $p \equiv 1 \pmod{3}$ . Denote by  $\mathcal{N}$  the norm function on  $\mathbb{Z}[\zeta_3]$ .

Let  $\pi$  be an irreducible element in  $\mathbb{Z}[\zeta_3]$ , and let  $v \in \mathbb{Z}[\zeta_3]$ . We define the *cubic* symbol  $\left(\frac{v}{\pi}\right)_3$  by the formulas

$$(2.1) \quad \left(\frac{v}{\pi}\right)_3 = \zeta_3^j, \pi \nmid v,$$

where  $j$  is the unique integer  $j \in \{0, 1, 2\}$  such that

$$v^{(N(\pi)-1)/3} \equiv \zeta_3^j \pmod{\pi}$$

and

$$\left(\frac{v}{\pi}\right)_3 = 0 \text{ if } \pi | v.$$

Observe that if  $\pi, \pi'$  are associates, then  $\left(\frac{\cdot}{\pi}\right)_3 = \left(\frac{\cdot}{\pi'}\right)_3$  and the function  $v \mapsto \left(\frac{v}{\pi}\right)_3$  is a multiplicative character of the group  $(\mathbb{Z}[\zeta_3]/\pi\mathbb{Z}[\zeta_3])^*$ . If  $q$  is a rational prime congruent to  $2 \pmod{3}$ , the restriction to  $\mathbb{Z}$  of the corresponding cubic character is simply the principal

character modulo  $q$ . We extend the definition of the cubic character to any element  $w \in \mathbb{Z}[\zeta_3]$  coprime to 3, by the formula

$$\left(\frac{v}{w}\right)_3 = \prod_j \left(\frac{v}{w_j}\right)_3,$$

where  $w = \prod_j w_j$  is the unique factorization (up to associates) of  $w$  into irreducible elements.

**2.2. Genus theory.** Let  $L/F$  be a cyclic extension of number fields, with  $\text{Gal}(L/F) = \langle \sigma \rangle$ . Write  $I_L$  to be the group of fractional ideals of  $L$  and  $P_L$  to be the group of principal fractional ideals of  $L$ , and similarly for  $F$ . There are two short exact sequences

$$0 \rightarrow \mathcal{O}_L^\times \rightarrow L^\times \rightarrow P_L \rightarrow 0 \text{ and}$$

$$0 \rightarrow P_L \rightarrow I_L \rightarrow \text{Cl}(L) \rightarrow 0.$$

Taking cohomology, and using the facts that  $H^1(\langle \sigma \rangle, L^\times) = H^1(\langle \sigma \rangle, I_L) = 0$ , we get three long exact sequences

$$0 \rightarrow \mathcal{O}_F^\times \rightarrow F^\times \rightarrow P_L[\sigma - 1] \rightarrow H^1(\langle \sigma \rangle, \mathcal{O}_L^\times) \rightarrow 0,$$

$$0 \rightarrow P_L[\sigma - 1] \rightarrow I_L[\sigma - 1] \rightarrow \text{Cl}(L)[\sigma - 1] \rightarrow H^1(\langle \sigma \rangle, P_L) \rightarrow 0, \text{ and}$$

$$0 \rightarrow H^1(\langle \sigma \rangle, P_L) \rightarrow H^2(\langle \sigma \rangle, \mathcal{O}_L^\times) \rightarrow H^2(\langle \sigma \rangle, L^\times).$$

The image of  $F^\times$  in  $P_L[\sigma - 1]$  is just  $P_F$ , so one gets  $H^1(\langle \sigma \rangle, \mathcal{O}_L^\times) = P_L[\sigma - 1]/P_F$ . Additionally, since  $\langle \sigma \rangle$  is a cyclic group, one knows that  $H^2(\langle \sigma \rangle, \mathcal{O}_L^\times) = \mathcal{O}_F^\times/N_{L/F}(\mathcal{O}_L^\times)$  and similarly for  $H^2(\langle \sigma \rangle, L^\times)$ . Thus, one can replace  $H^2(\langle \sigma \rangle, \mathcal{O}_L^\times) \rightarrow H^2(\langle \sigma \rangle, L^\times)$  with  $H^2(\langle \sigma \rangle, \mathcal{O}_L^\times) \rightarrow \mathcal{O}_F^\times/(N_{L/F}(F^\times) \cap \mathcal{O}_F^\times) \rightarrow 0$ . Dividing the first two terms in the second sequence by  $P_F$ , we can stitch everything together into one sequence:

$$0 \rightarrow H^1(\langle \sigma \rangle, \mathcal{O}_L^\times) \rightarrow I_L[\sigma - 1]/P_F \rightarrow \text{Cl}(L)[\sigma - 1] \rightarrow H^2(\langle \sigma \rangle, \mathcal{O}_L^\times) \rightarrow \mathcal{O}_F^\times/(N_{L/F}(F^\times) \cap \mathcal{O}_F^\times) \rightarrow 0.$$

More detailed analysis is possible in general, but at this point, we find it useful to switch to the case that  $F = K (= \mathbb{Q}(\zeta_3))$  and  $L/K$  is cyclic and degree 3. Kummer theory says that  $L = K(\sqrt[3]{\alpha})$  for some  $\alpha \in K$ . Every ideal of  $K$  is principal, so the second term in the long exact sequence is  $I_L[\sigma - 1]/I_K$ . Now, we can write  $I_L = \bigoplus_{\mathfrak{p} \subset \mathcal{O}_K} \bigoplus_{\mathfrak{p} | \mathfrak{p} \mathcal{O}_L} \mathfrak{p}^{\mathbb{Z}}$ . If  $\mathfrak{p}$  splits or is inert in  $L$ , then a straightforward calculation shows that  $(\bigoplus_{\mathfrak{p} | \mathfrak{p} \mathcal{O}_L} \mathfrak{p}^{\mathbb{Z}})[\sigma - 1] = \mathfrak{p}^{\mathbb{Z}}$ , and if  $\mathfrak{p}$  is ramified in  $L$ , then it's easy to check that  $(\bigoplus_{\mathfrak{p} | \mathfrak{p} \mathcal{O}_L} \mathfrak{p}^{\mathbb{Z}})[\sigma - 1] = \mathfrak{p}^{\frac{1}{3}\mathbb{Z}}$ . Thus, the second term becomes  $\bigoplus_{\mathfrak{p} \text{ ramified in } L} \mathfrak{p}^{\frac{1}{3}\mathbb{Z}/\mathbb{Z}}$ . The question that we are interested in is whether there is an element  $u \in \mathcal{O}_L^\times$  such that  $N_{L/K}(u) = \zeta_3$ , but that is exactly equivalent to  $H^2(\langle \sigma \rangle, \mathcal{O}_L^\times) = 0$ , which then can be seen to be equivalent to the two statements that  $\mathcal{O}_F^\times/(N_{L/F}(F^\times) \cap \mathcal{O}_F^\times) = 0$  and that  $\text{Cl}(L)[\sigma - 1]$  is generated by the ramified primes in  $L/K$ . At this point, we are prepared to produce the first algebraic criterion for the  $\zeta_3$ -Pell equation to have a solution.

**Theorem 2.1.** *Write  $L = K(\sqrt[3]{\alpha})$  as above, and assume that  $\alpha = \pi_1^{a_1} \cdots \pi_n^{a_n}$  with  $\pi_i \equiv 1 \pmod{\lambda^2}$  and  $a_i = 1$  or  $2$ . Then there is a solution to the  $\zeta_3$ -Pell equation if and only if all the  $\pi_i$ s satisfy  $\pi_i \equiv 1 \pmod{\lambda^3}$  and the primes  $\mathfrak{P}_i$  lying above  $\pi_i \mathcal{O}_K$  generate  $\text{Cl}(L)[\sigma - 1]$ .*

*Proof.* Based on the discussion above, what is needed to be shown is that  $\alpha$  has the specified form if and only if  $\mathcal{O}_F^\times/(N_{L/F}(F^\times) \cap \mathcal{O}_F^\times) = 0$ .

If  $\alpha$  is divisible by a prime  $\pi$  with  $\pi \not\equiv 1 \pmod{\lambda^3}$  then, writing  $K_\pi$  for the completion of  $K$  at  $\pi$  and  $L_\pi$  for  $L \otimes_K K_\pi$ , one has that  $L_\pi$  is a ramified cubic extension of  $K_\pi$ . Thus,  $N_{L_\pi/K_\pi}(\mathcal{O}_{L_\pi}^\times)$  is the unique index three subgroup of  $\mathcal{O}_{K_\pi}^\times$ . Since  $\pi \not\equiv 1 \pmod{\lambda^3}$ , there is no ninth root of unity in  $K_\pi$ , and so  $\zeta_3$  isn't a cube in  $K_\pi$  and consequently not a norm

from  $\mathcal{O}_{L^\times}$ . Thus, if the first condition is not satisfied, then there is a local obstruction to  $\zeta_3$  being a norm of an element of  $L^\times$  and so  $\mathcal{O}_F^\times / (N_{L/F}(F^\times) \cap \mathcal{O}_F^\times) \neq 0$ .

As for the other direction, assume that  $\alpha$  is of the form described. Classical calculations (see for example (tbc)) show that  $h_{2/1}(\langle \sigma \rangle, \mathcal{O}_L^\times) = 1/3$ , and thus one gets that  $\dim_{\mathbb{F}_3}(\text{Cl}(L)[\sigma - 1]) = n - 1 - \dim_{\mathbb{F}_3}(\mathcal{O}_F^\times / (N_{L/F}(F^\times) \cap \mathcal{O}_F^\times))$  as there are  $n$  primes ramified in  $L/K$ . But one also has that  $\dim_{\mathbb{F}_3}(\text{Cl}(L)[\sigma - 1]) = \dim_{\mathbb{F}_3}(\text{Cl}(L)/(\sigma - 1))$ . Moreover,  $L(\sqrt[3]{\pi_1}, \dots, \sqrt[3]{\pi_{n-1}})$  is a degree  $3^{n-1}$  extension of  $L$  that is unramified everywhere (the total extension is unramified at  $\lambda$  because all of cuberoots are of elements that are  $1 \pmod{\lambda^3}$ ). Further, it is abelian over  $K$  and thus corresponds to a  $\sigma - 1$  invariant quotient of  $\text{Cl}(L)$ . One has then that  $n - 1 \leq \dim_{\mathbb{F}_3}(\text{Cl}(L)[\sigma - 1]) = n - 1 - \dim_{\mathbb{F}_3}(\mathcal{O}_F^\times / (N_{L/F}(F^\times) \cap \mathcal{O}_F^\times))$  and so  $\mathcal{O}_F^\times / (N_{L/F}(F^\times) \cap \mathcal{O}_F^\times) = 0$ , which is what was needed.  $\square$

The proof of this theorem also proves the following:

**Corollary 2.2.** *With  $L$ ,  $\alpha$ , and  $\pi_i$  as above, and assuming that  $\pi_i \equiv 1 \pmod{\lambda^3}$  for all  $i$ , one has that the maximal unramified  $\sigma - 1$  cotorsion extension of  $L$  is  $L(\sqrt[3]{\pi_1}, \dots, \sqrt[3]{\pi_{n-1}})$ .*

**2.3. Rédei matrices.** Fix  $L = K(\sqrt[3]{\alpha})$ , and assume that  $\alpha = \pi_1^{a_1} \dots \pi_n^{a_n}$  with  $\pi_i \equiv 1 \pmod{\lambda^3}$ . The goal now is to understand the group  $\text{Cl}(L)_{\sigma-1}$ , which is all of the  $(\sigma - 1)$ -power torsion in the class group. We will not come to a complete description, but we will be able to say more than just what was said in the previous subsection.

The main trick is the following: if  $M$  is an  $R_\sigma = \mathbb{Z}[\sigma]/\langle \sigma^2 + \sigma + 1 \rangle$ -module which is finite as a group, then one may consider the map from  $M[\sigma - 1] \rightarrow M/(\sigma - 1)$ . This is a map from  $\mathbb{F}_3$ -vector spaces of the same dimension, and so is given by a matrix. One can compute this rank by appealing to the fact that every  $R_\sigma$ -module that is finite as a group is of the form  $\bigoplus_{\pi_i} \bigoplus_{a_{ij}} R_\sigma / \pi_i^{a_{ij}}$  where the first sum runs over some primes in  $R_\sigma$  and  $a_{ij}$ s are positive integers. Assuming that  $\pi_1$  is the prime  $\sigma - 1$ , then one has that the dimension of the spaces  $M[\sigma - 1]$  and  $M/(\sigma - 1)$  are the number of  $a_{1j}$ s, and the rank is the number of  $a_{1j}$ s that are equal to one. Finally, it is actually useful to think of this not as a map but rather as a pairing  $M[\sigma - 1] \times M^\vee[\sigma - 1] \rightarrow \mathbb{F}_3$ , where now the left kernel of this pairing is  $(\sigma - 1)M[(\sigma - 1)^2]$ . This is obviously equivalent, but in the case of class groups, its simpler to write down the pairing than the actual map.

To apply this idea to the setup here, let  $\mathfrak{P}_i$  be the prime lying over  $\pi_i$ . While it is not clear if the primes  $\mathfrak{P}_i$  generate  $\text{Cl}(L)[\sigma - 1]$ , we can still consider the subgroup of  $\text{Cl}(L)[\sigma - 1]$  generated by these primes. There is the relation  $\mathfrak{P}_1^{a_1} \dots \mathfrak{P}_n^{a_n} = \sqrt[3]{\alpha} \mathcal{O}_L$  which says that we only need to consider the subgroup generated by  $\mathfrak{P}_i$  for  $i$  running from 1 to  $n - 1$ . Additionally, we know by corollary 2.2 that  $\text{Cl}(L)/(\sigma - 1) = \text{Gal}(L(\sqrt[3]{\pi_1}, \dots, \sqrt[3]{\pi_{n-1}})/L)$ . Thus, one has that  $\text{Cl}(L)^\vee[\sigma - 1]$  is generated by characters  $\chi_i$  with  $\chi_i(\mathfrak{P}_j) = \left(\frac{\pi_j}{\pi_i}\right)$  for  $i \neq j$  and by using the fact that  $\chi_i(\mathfrak{P}_1^{a_1} \dots \mathfrak{P}_n^{a_n}) = 1$  to compute  $\chi_i(\mathfrak{P}_i)$ .

To write down the actual matrix, we will write  $\log\left(\left(\frac{\beta}{\pi}\right)_3\right) = a$  for  $a \in \mathbb{F}_3$  if  $\left(\frac{\beta}{\pi}\right)_3 = \zeta_3^a$ . Then if we let  $b_{ii} = a_i^{-1} \log\left(\left(\frac{\pi_i^{a_i}/\alpha}{\pi_i}\right)_3\right)$  and  $b_{ij} = \log\left(\left(\frac{\pi_i}{\pi_j}\right)_3\right)$ , one has that the matrix  $M_L = (b_{ij})$  represents the pairing the subgroup of  $\text{Cl}(L)[\sigma - 1]$  generated by the primes  $\mathfrak{P}_i$  with  $\text{Cl}(L)^\vee[\sigma - 1]$ .

**Theorem 2.3.** *Assume that  $M_L$  has full rank. Then there is a unit  $u \in \mathcal{O}_L^\times$  such that  $N_{L/K}(u) = \zeta_3$ .*

*Proof.* Let  $V$  be the free  $n - 1$ -dimensional  $\mathbb{F}_3$  vector space with basis vectors  $e_i$ . Saying  $M_L$  has full rank is the same thing as saying that the composite map  $V \rightarrow \text{Cl}(L)[\sigma - 1] \rightarrow \text{Cl}(L)/(\sigma - 1)$  has full rank (where the first map sends  $e_i$  to  $\mathfrak{P}_i^{a_i}$  and the second map is the natural one). But these are all maps between  $n - 1$ -dimensional vector spaces over  $\mathbb{F}_3$  and so if the composite map has full rank, so too does the first one. But that just says that the ramified primes generate  $\text{Cl}(L)[\sigma - 1]$ , which by theorem 2.1 shows that there is a unit  $u \in \mathcal{O}_L^\times$  such that  $N_{L/K}(u) = \zeta_3$ .  $\square$

This condition is not necessary for the  $\zeta_3$ -Pell equation to be solvable. Indeed, playing around with fields of the form  $K(\sqrt[3]{\pi_1\pi_2})$ , one sees that roughly 2/3s of such fields satisfy the condition, but an additional roughly 1/36 don't satisfy the condition but nevertheless have a solution to the  $\zeta_3$ -Pell equation.

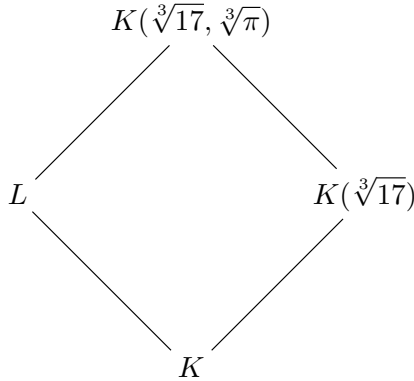
### 3. GOVERNORING FIELDS

The goal of this section is to produce a negative criterion for the  $\zeta_3$ -Pell equation, allowing us to bound above the number of fields which solve the equation.

**Theorem 3.1.** *Keeping notation as above, if one has that  $\dim(\ker(M_L)) > \dim((\sigma - 1)\text{Cl}(L)/(\sigma - 1)^2)$ , then there is not a solution to the  $\zeta_3$ -Pell equation.*

*Proof.* As discussed above, if there is a solution to the  $\zeta_3$ -Pell equation, one has that  $M_L$  is the actual R edei pairing and not just the pairing restricted to a subgroup of the  $\sigma - 1$ -torsion. Thus, one would have  $\dim(\ker(M_L)) = \dim((\sigma - 1)\text{Cl}(L)/(\sigma - 1)^2)$  in this case, contradicting the assumption in the theorem.  $\square$

Before constructing the governing fields in full generality, we find it nice to focus on one simple case. We will look at fields of the form  $L = K(\sqrt[3]{17\pi})$ . Observe that there is a diagram of fields with the field corresponding to  $\text{Cl}(L)/(\sigma - 1)$  on top:



As discussed before, the story is over if  $(\frac{\pi}{17})_3 \neq 1$ : under this assumption we have that all of the  $\sigma - 1$ -power part of the class group is  $\sigma - 1$ -torsion. Thus, we will assume that  $(\frac{\pi}{17})_3 = 1$ . We want to know if there is an unramified extension  $L_2/L$  such that  $\text{Gal}(L_2/L) = R_\sigma/(\sigma - 1)^2$ . Writing  $F = K(\sqrt[3]{17\pi})$  and  $\tau$  for the generator of  $\text{Gal}(F/K)$ , we have any such field is also abelian over  $F$  with Galois group  $R_\tau/(\tau - 1)^2$ , ramified only at  $\pi$ . But now class field theory intervenes and we can compute the maximal extension of  $F$ , unramified away from  $\pi$ , with Galois group killed by  $\tau - 1$ . Denoting this field by  $F_\pi$ , we have that  $\text{Gal}(F_\pi/F)$  is surjected onto by  $V_\pi := (\mathcal{O}_F/\pi)^\times \otimes_{\mathbb{Z}} \mathbb{F}_3$ , and the kernel of this surjection is generated by  $V_\pi[\tau - 1]$  and the image of  $\mathcal{O}_L^\times$ . Thus, there is a field  $L_2/F$  if and only if  $\mathcal{O}_L^\times \subset V_\pi[\tau - 1]$ , which happens if and only if  $(\mathcal{O}_F^\times)^{\tau-1}$  all reduce to cubes mod  $\pi$ . But that is tantamount to saying that  $\pi$  splits in  $F(\sqrt[3]{(\mathcal{O}_F^\times)^{\tau-1}})$ . One can easily compute that  $\zeta_9$  is in this field, and that the degree of the extension  $F(\sqrt[3]{(\mathcal{O}_F^\times)^{\tau-1}})/F(\zeta_9)$  is 3. Thus, one

gets that, among all primes  $\pi \equiv 1 \pmod{\lambda^3}$ , two-thirds of them don't have  $(\frac{\pi}{17})_3 = 1$  and thus have a solution to the  $\zeta_3$ -Pell equation, two-thirds of the remainder do have  $(\frac{\pi}{17})_3 = 1$  but don't split in  $F(\sqrt[3]{(\mathcal{O}_F^\times)^{\tau-1}})$  and so don't have a solution to the  $\zeta_3$ -Pell equation. The authors as of right now have no ideas for how to get deeper into this set.

To set up the full generality, we will consider fields of the form  $L_\pi = K(\sqrt[3]{\alpha\pi})$  with  $\alpha = \pi_1^{a_1} \cdots \pi_n^{a_n}$ . Let  $F = K(\sqrt[3]{\pi_1}, \dots, \sqrt[3]{\pi_n})$ , and write  $\text{Gal}(F/K) = \langle \tau_1, \dots, \tau_n \rangle$  with  $\tau_i(\sqrt[3]{\pi_i}) = \zeta_3 \sqrt[3]{\pi_i}$  and  $\tau_i(\sqrt[3]{\pi_j}) = \sqrt[3]{\pi_j}$  for  $i \neq j$ . Choose a matrix  $M = (b_{ij})$  such that  $b_{ij} = \log\left(\left(\frac{\pi_i}{\pi_j}\right)_3\right)$  for  $i \neq j$ . Writing  $M_\pi$  for the matrix associated to the pairing as constructed in subsection 2.3, one has that there is an element  $\tau$  in  $\text{Gal}(F/K)$  such that  $\text{Frob}_\pi = \tau$  if and only if  $M_\tau = M$ . The main theorem is as follows:

**Theorem 3.2.** *There is a field  $F_\alpha/F(\zeta_9)$  of degree  $\dim(\ker(M))$  such that  $F_\alpha$  is Galois over  $K$ , and, for all  $\pi$  with  $M_\pi = M$ , one (equivalently any) prime  $\mathfrak{P}$  lying over  $\pi$  in  $F$  splits in  $F_\alpha$  if and only if  $\dim((\sigma - 1)\text{Cl}(L_\pi)[(\sigma - 1)^2]) = \dim(\ker(M))$ .*

*Proof.* What follows will be essentially the same argument, with a lot more notation. Let  $r_1 : \mathbb{F}_3\langle e_1, \dots, e_n \rangle \rightarrow \text{Cl}(L)/(\sigma - 1)$  sending  $e_i$  to the class associated to the prime over  $\pi_i$ . The assumptions on  $\pi$  imply that the cokernel of  $r_1$  is independent of  $i$ , and so we may view  $W := \text{coker}(r_1)$  as a fixed subgroup of  $\text{Gal}(F/K)$ . Let  $R = \mathbb{F}_3 + W$ , a ring where one defines multiplication by setting  $\tau\tau' = 0$  for any two  $\tau, \tau' \in W$ .  $R$  is naturally a quotient of the group ring  $\mathbb{F}_3[\text{Gal}(F/K)]$ .

Define  $V_\pi := (\mathcal{O}_F/\pi)^\times \otimes_{\mathbb{Z}} \mathbb{F}_3$  and  $V'_\pi = V_\pi \otimes_{\mathbb{F}_3[\text{Gal}(F/K)]} R$ . The assumptions on  $\pi$  imply that  $V'_\pi$  is isomorphic to  $R$  as an  $R$ -module. Write  $L_2$  to be the maximal abelian extension of  $L$  unramified everywhere and whose Galois group is  $(\sigma - 1)^2$ -torsion. One has then that  $L_2/F$  is abelian and unramified outside of  $\pi$ , and  $\text{Gal}(L_2/F)$  is an  $R$ -module. Class field theory gives a map from  $V'_\pi \rightarrow \text{Gal}(L_2/F)$  which is surjective, and whose kernel is the image of global units. However, one has that  $\dim_{\mathbb{F}_3}(\text{Gal}(L_2/F)) = \dim((\sigma - 1)\text{Cl}(L_\pi)[(\sigma - 1)^2]) + 1$ , so one has  $\dim((\sigma - 1)\text{Cl}(L_\pi)[(\sigma - 1)^2]) = \dim(\ker(M))$  if and only if there is no kernel in the map from  $V'_\pi \rightarrow \text{Gal}(L_2/F)$ . Asking that the image of  $\mathcal{O}_F^\times$  in  $V'_\pi$  is trivial is just asking that  $r \cdot u$  is a cube for all  $r \in \ker(\mathbb{F}_3[\text{Gal}(F/K)] \rightarrow R)$  and  $u \in \mathcal{O}_F^\times$ . This gives the existence of the field  $F_\alpha/F$  by taking the corresponding Kummer extension; all that's left is to compute the degree.

Now, the structure of  $\mathcal{O}_F^\times \otimes_{\mathbb{Z}} \mathbb{F}_3 = \mathfrak{m}$  as an  $\mathbb{F}_3[\text{Gal}(F/K)]$ -module where  $\mathfrak{m}$  is the unique maximal ideal in  $\mathbb{F}_3[\text{Gal}(F/K)]$ . Asking that  $(\mathcal{O}_F^\times \otimes_{\mathbb{Z}} \mathbb{F}_3) \otimes_{\mathbb{F}_3[\text{Gal}(F/K)]} R$  maps to 0 in  $V'_\pi$  is imposing  $\dim(W)$  different conditions, as  $\mathfrak{m} \otimes_{\mathbb{F}_3[\text{Gal}(F/K)]} R = W$ . That gives the degree of  $F_\alpha$  and completes the proof of the theorem.  $\square$

At this point, one can imagine how the proof of the main theorem goes. Firstly, one wants to show that the matrices  $M_L$  look like large random symmetric matrices over  $\mathbb{F}_3$ . This gives the lower bound on the number of fields that solve the  $\zeta_3$ -Pell equation. This also gives a distribution on the dimension of the kernels of the matrices  $M_L$ , and the next step is to show that for fields  $L$  with  $\dim(\ker(M_L)) = d$ , there is a  $\frac{3^d - 1}{3^d}$  chance that Theorem 3.1 applies. This gives the upper bound.

#### 4. DISTRIBUTION OF THE 3-RANK

This section splits up into two parts. First, we compute the ranks of the matrices  $M_L$  for varying  $L$ , and then we use that to compute the dimensions of  $(\sigma - 1)(\text{Cl}(L)[(\sigma - 1)^2])$  for varying  $L$ .

**4.1. Ranks of Matrices.** For an element  $s \in \mathbb{Z}[\zeta_3]$ , we denote by  $N(s)$  its norm. We will require the following lemma regarding cubic character sums:

**Lemma 4.1.** *Let  $\{\alpha_m\}, \{\beta_n\}$  be two complex sequences indexed by the Eisenstein integers such that  $|\alpha_n|, |\beta_m| \leq 1$  for all  $n, m \in \mathbb{N}$ . Let  $\mu$  be the natural extension of the Mobius function to  $\mathbb{Z}[\zeta_3]$ . Put*

$$\Xi(M, N, \alpha, \beta) = \sum_{N(m) \leq M} \sum_{N(n) \leq N} \alpha_m \beta_n \mu^2(m) \mu^2(n) \left(\frac{m}{n}\right)_3,$$

and

$$S_1(M, N) = N^{-1/2} + M^{-1/4} N^{1/2}, S_2(M, N, \varepsilon) = M^\varepsilon \left( N^{-1/8} + M^{-1/4} N^{1/8} \right).$$

Then for all  $\varepsilon > 0$

$$(4.1) \quad \Xi(M, N, \alpha, \beta) = O_\varepsilon(MN \min \{S_1(M, N), S_1(N, M), S_2(M, N, \varepsilon), S_2(N, M, \varepsilon)\}).$$

*Proof.* See Proposition 9 in [5] and Remark 6.3, as well as [2].  $\square$

We put

$$(4.2) \quad S(X) = \{L = K(\sqrt[3]{\alpha}) : |\Delta_L| \leq X, \pi|\alpha \Rightarrow N(\pi) \equiv 1 \pmod{9}\}$$

and for  $n \geq 1, n \in \mathbb{Z}$ ,

$$(4.3) \quad S_n(X) = \{L \in S(X) : \dim \ker M_L = n\}.$$

We wish to show that the limit

$$\lim_{X \rightarrow \infty} \frac{|S_n(X)|}{|S(X)|}$$

exists and is equal to the corresponding limit for random  $\mathbb{F}_3$ -matrices. Precisely, one defines  $\beta_n := \frac{\beta}{\prod_{j=1}^n (3^j - 1)}$ , and we wish to show

**Theorem 4.2.** *One has that*

$$\lim_{X \rightarrow \infty} \frac{|S_n(X)|}{|S(X)|} = \beta_n.$$

That  $\beta_n$  gives the right distribution is classical, and can be seen directly from Proposition 9 in [9]. To wit, we decompose  $S_n(X)$  as

$$\begin{aligned} S_n(X) &= \bigcup_{r \geq 0} \{L : |\Delta_L| \leq X, \dim \ker M_L = n, \omega(\alpha) = r + 1\} \\ &= \bigcup_{r \geq 0} S_n^{(r)}(X) \\ &= \bigcup_{r \geq 0} \bigcup_{\substack{M \text{ symmetric } r \times r \text{ } \mathbb{F}_3 \text{ matrix} \\ \text{rk } M = n - r}} \{L : |\Delta_L| \leq X, M_L = M, \omega(\alpha) = r + 1\}. \end{aligned}$$

We denote by  $S_n^{(r)}(M; X)$  a term in the last line.

We wish to write down the condition  $M_L = M$  in terms of cubic symbols. For a fixed  $a_{ij} \in \mathbb{F}_3$ , we have that the indicator function for  $a_{ij} = m_{ij}$  is given by

$$(4.4) \quad u_{ij} = \frac{1 + \zeta_3^{-a_{ij}} \left(\frac{\pi_i}{\pi_j}\right)_3 + \zeta_3^{-2a_{ij}} \left(\frac{\pi_i}{\pi_j}\right)_3^2}{3}.$$

We may assume going forward that the number of prime factors is bounded by

$$(4.5) \quad \log \log X - (\log \log X)^{3/4} < r < \log \log X + (\log \log X)^{3/4}.$$

This follows from an appropriate application of the Selberg-Delange method, and a corresponding refinement to Theorem 5 in [10]. From here, we shall estimate  $S_n^{(r)}(A; X)$  individually, being careful to produce error terms that depend explicitly on  $r$  and otherwise uniform.

We put

$$A = (a_{ij}), a_{ij} \in \mathbb{F}_3, a_{ij} = a_{ji}, 1 \leq i, j \leq r$$

for a fixed  $r \times r$  symmetric  $\mathbb{F}_3$ -matrix. Recall from (4.4) that we can write

$$\text{Ind}(M_L = A) = \prod_{i \leq j} u_{ij}.$$

Expanding, we see that we are left to deal with sums of the shape

$$(4.6) \quad \sum_{N(\pi_1 \cdots \pi_r) \leq X} 3^{-\frac{r^2+r}{2}} \prod_{i \leq j} \left( \frac{\pi_i}{\pi_j} \right)_3^{b_{ij}},$$

where  $b_{ij} \in \{0, 1, 2\}$ . Observe that the term with  $b_{ij} = 0$  for all  $i, j$  is expected to be the main term.

Put

$$(4.7) \quad T_r(D; X) = \{\alpha = \pi_1 \cdots \pi_r \leq X : N(\pi_i) > D, N(\pi_i) \equiv 1 \pmod{9} \text{ for } 1 \leq i \leq r\}.$$

In other words,  $T_r(D; X)$  is the set of elements in  $\mathcal{O}_K$  having exactly  $r$  prime factors (up to multiplicity) and norm bounded by  $X$ , such that the norm of each prime factor exceeds  $D$  and is congruent to 1 (mod 9).

For an element  $n = p_1 \cdots p_r \in T_r(D; X)$ , put

$$d_{2,r} = d_{2,r}(n) = \prod_{2 \leq i \leq r} N(\pi_i).$$

Let  $\mathcal{T}_r(D; X)$  denote the subset of  $T_r(D; X)$  having the property that

$$d_{2,r} > X e^{-\exp(\sqrt{\log \log X})}.$$

We require:

**Lemma 4.3.** *Let  $\sum_r^\dagger$  denote summing over the range*

$$\log \log X - (\log \log X)^{3/4} < r < \log \log X + (\log \log X)^{3/4}.$$

*Then*

$$\lim_{N \rightarrow \infty} \frac{\sum_r^\dagger |T_r(D; X)|}{\sum_r^\dagger |T_r(D; X)|} = 0.$$

*Proof.* By the arguments given in Section 5 of [7], we have the estimate

$$k_1 \frac{rX(\log \log X)^{r-1}}{6^r \log X} < |T_r(D; X)| < k_2 \frac{rX(\log \log X)^{r-1}}{6^r \log X}$$

for some positive numbers  $k_1, k_2$  and sufficiently large  $X$ . Indeed, we can even take  $D$  to be a slowly increasing function of  $N$ ; taking  $D = \log \log \log X$  suffices.

Using the bound

$$|T_r(D; X)| \leq \sum_{\substack{p_1=1 \\ p_1 \equiv 1 \pmod{9}}}^{\exp(\exp(\sqrt{\log \log X}))} T_{r-1}(D; X),$$



we then see that

$$(4.8) \quad |\mathcal{T}_r(D; X)| \leq \sum_{p_1} \frac{k_2 r (X/p_1) (\log \log X)^{r-2}}{6^r \log X} \ll \frac{r X (\log \log X)^{r-2}}{6^r \log X} \sum_{p_1}^{\exp(\exp(\sqrt{\log \log X}))} \frac{1}{p_1}.$$

By Dirchlet's theorem applied to primes congruent to 1 mod 9, we find that

$$\sum_{\substack{p_1=1 \\ p_1 \equiv 1 \pmod{9}}}^{\exp(\exp(\sqrt{\log \log X}))} \frac{1}{p_1} = \frac{(\log \log X)^{1/2}}{6} + B_0 + O((\log X)^{-1}),$$

where  $B_0$  is an absolute constant. It therefore follows that

$$(4.9) \quad |\mathcal{T}_r(D; X)| = O\left(r 6^{-r} X (\log \log X)^{r-3/2} (\log X)^{-1}\right),$$

and thus we see that

$$\frac{|\mathcal{T}_r(D; X)|}{|T_r(D; X)|} = O\left((\log \log X)^{-1/2}\right),$$

which tends to 0 as  $X \rightarrow \infty$ , as desired.  $\square$

We may thus restrict our attention to the set:

$$(4.10) \quad T_r^*(D; X) = \left\{ m \in T_r(D; X) : p|m \Rightarrow p > \exp\left(\exp\left(\sqrt{\log \log X}\right)\right) \right\}.$$

Put

$$(4.11) \quad X^\dagger = \exp\left(\exp\left(\sqrt{\log \log X}\right)\right),$$

and consider the contribution from those sums (4.6) such that there exist two indices  $u, v$  for which  $b_{uv} \in \{1, 2\}$  and such that  $B_u, B_v \gg X^\dagger$ . Denote this subset by  $S^\dagger(\mathbf{B}; X)$ . Put

$$(4.12) \quad \Xi(u, v) = \prod_{w \neq u, v} \left( \frac{\pi_u}{\pi_w} \right)_3^{b_{uw}}.$$

We then have the bound

$$(4.13) \quad \left| S^\dagger(\mathbf{B}; X) \right| \leq \sum_{\pi_w: w \neq u, v} \prod_{w \neq u, v} 3^{-(r^2+r)/2} \left| \sum_{\pi_u} \sum_{\pi_v} \Xi(u, v) \Xi(v, u) \left( \frac{\pi_u}{\pi_v} \right)_3^{b_{uv}} \right|,$$

and  $b_{uv} \in \{1, 2\}$ . The upshot is that  $|\Xi(u, v)| \leq 1$  so Lemma 4.1 applies. We thus obtain the bound, for all  $\varepsilon > 0$ ,

$$\left| S^\dagger(\mathbf{B}; X) \right| \ll_\varepsilon \left( \prod_{w \neq u, v} B_w \right) \left( B_u B_v \left( B_u^{-1/8+\varepsilon} + B_v^{-1/8+\varepsilon} \right) \right) \ll_\varepsilon X \left( X^\dagger \right)^{-1/8+\varepsilon},$$

which gives the bound

$$(4.14) \quad \left| S^\dagger(\mathbf{B}; X) \right| \ll X \exp\left(-c \exp\left(\sqrt{\log \log X}\right)\right)$$

for some  $c > 0$ . This is smaller than  $X(\log X)^{-A}$  for any  $A > 0$ , which is enough for us.

We have thus shown that all sums of the form (4.6) with at least one pair of indices  $\{i, j\}$  with  $b_{ij} \neq 0$  contributes a negligible amount to  $S_n^{(r)}(A; X)$ , and hence only the main term where  $b_{ij} = 0$  for all  $i, j$  contributes.

**4.2. Dimensions of Parts of Class Groups.** Our goal is to translate Theorem 3.2 into a statement about the distribution of 3-ranks of  $L = K(\sqrt[3]{\alpha})$  over  $K$ , as  $\alpha$  ranges over  $\mathcal{O}_K$  with bounded norm.

We first extract a probabilistic consequence of Theorem 3.2:

**Proposition 4.4.** *Let  $\alpha \in \mathcal{O}_K$  with  $\alpha = \pi_1 \cdots \pi_r$ . Put  $L_\pi = K(\sqrt[3]{\alpha\pi})$ . Fix a symmetric  $r \times r$   $\mathbb{F}_3$ -matrix  $M = (m_{ij})$ , and suppose that*

$$m_{ij} = \log \left( \frac{\pi_i}{\pi_j} \right)_3 \quad \text{for } i \neq j.$$

*Fix a coset  $c \in (\mathcal{O}_K/(\alpha))^*$  such that  $\pi \in c$  if and only if  $M_{L_\pi} = M$ . Let  $m = \dim \ker(M)$ . Let  $\mathcal{L}_c(X)$  be the set of primes  $\pi \in \mathcal{O}_K$  with  $N(\pi) \leq X$  and  $N(\pi) \equiv 1 \pmod{9}$ , with  $\pi \in c$ . Then*

$$\lim_{X \rightarrow \infty} \frac{|\{\pi \in \mathcal{L}_c(X) : \dim(\sigma - 1) \text{Cl}(L_\pi)[(\sigma - 1)^2] = m\}|}{|\mathcal{L}_c(X)|} = 3^{-m}.$$

*Likewise,*

$$\lim_{X \rightarrow \infty} \frac{|\{\pi \in \mathcal{L}_c(X) : \dim(\sigma - 1) \text{Cl}(L_\pi)[(\sigma - 1)^2] \neq m\}|}{|\mathcal{L}_c(X)|} = 1 - 3^{-m}.$$

*Proof.* This follows from Theorem 3.2 and Chebotarev's density theorem applied to the field  $F(\alpha)$ .  $\square$

In order to obtain the analogous result where we allow more variation than one prime at a time, we follow the strategy in [6]; in particular, a stronger form of Chebotarev's density theorem is needed. To obtain such a version we simply apply GRH.

**Proposition 4.5** (Effective Chebotarev's density theorem conditioned on GRH). *Let  $L/K$  be a finite Galois extension of number fields. Let  $C \subset \text{Gal}(L/K)$  be a union of conjugacy classes and*

$$\psi(C, X) = \sum_{\substack{N(\mathfrak{p}) \leq X \\ \text{Frob}_{\mathfrak{p}} \in C}} \log N(\mathfrak{p}).$$

*If GRH holds for Artin L-functions, then for  $X \geq 2$*

$$\psi(C, X) = \frac{|C|X}{|\text{Gal}(L/K)|} + O\left(X^{1/2} \sum_{\rho} |c_{\rho}| \log\left(X^{\deg(\rho)[K:\mathbb{Q}]_q(\rho)}\right)\right)$$

*where the sum runs over all irreducible representations of  $\text{Gal}(L/K)$ ,  $c_{\rho}$  is given by*

$$c_{\rho} = \frac{1}{|\text{Gal}(L/K)|} \sum_{x \in C} \overline{\text{Tr}(\rho(x))},$$

*and  $q(\rho)$  is the conductor.*

In order to apply Proposition 4.5, we will need to bound the conductor of  $F(\alpha)$ . This follows from the fact that  $F(\alpha)$  is unramified over  $K(\sqrt[3]{\alpha})$  and degree  $3^{r+m}$  over  $K(\sqrt[3]{\alpha})$ . In particular, the conductor of  $F(\alpha)$  is at most

$$N(\alpha)^{3^{r+m}}.$$

Applying Proposition 4.5 to Proposition 4.4 then gives

$$(4.15) \quad \sum_{\pi \in \mathcal{L}_c(X^\dagger)} 1 = \frac{\text{Li}(X^\dagger)}{3^{r+1}} + O\left((r+m)(X^\dagger)^{1/2} \log X^\dagger\right)$$

and

$$(4.16) \quad \sum_{\substack{\pi \in \mathcal{L}_c(X^\dagger) \\ \dim(\sigma-1)\text{Cl}(L_\pi)[(\sigma-1)^2]=m}} 1 = \frac{\text{Li}(X^\dagger)}{3^{m+r+1}} + O\left((r+m)(X^\dagger)^{1/2} \log X^\dagger\right).$$

Both follows from Proposition 4.5. Note that the error terms in (4.15) and (4.16) are independent of  $\alpha$ , whence we may sum both uniformly across  $\alpha$  to obtain the following:

**Theorem 4.6.** *Suppose that GRH holds for Dedekind zeta functions, so that Proposition 4.5 holds. Then*

$$\lim_{X \rightarrow \infty} \frac{|\{\alpha \in S_m(X) : \dim(\sigma-1)\text{Cl}(K(\sqrt[3]{\alpha})[(\sigma-1)^2] = m)\}|}{|S_m(X)|} = 3^{-m}.$$

We will find it useful to name the set  $\{\alpha \in S_m(X) : \dim(\sigma-1)\text{Cl}(K(\sqrt[3]{\alpha})[(\sigma-1)^2] = m\} := S'_m(X)$ .

## 5. PROOF OF THEOREM 1.1 AND HEURISTICS BEHIND CONJECTURE 1.2

Now, we put together all of the previous discussion into a proof of Theorem 1.1.

*Proof of Theorem 1.1.* The upshot of the discussion in Section 4 is that

$$\lim_{X \rightarrow \infty} \frac{|S_m(X)|}{|S(X)|} = \beta_m,$$

and

$$\lim_{X \rightarrow \infty} \frac{|S'_m(X)|}{|S(X)|} = \frac{\beta_m}{3^m}.$$

If  $\alpha \in S_0(X)$ , then the  $\zeta_3$ -Pell equation for  $K(\sqrt[3]{\alpha})$  has a solution by Theorem 2.3, and if  $K(\sqrt[3]{\alpha})$  has a solution to the  $\zeta_3$ -Pell equation, then  $\alpha \in S'_m(X)$  for some  $m$  by Theorem 3.1. One clearly has  $\beta_0 = \beta$ , which gives the lower bound in Theorem 1.1. To get the upper bound, we need to compute the sum  $\sum_{m=0}^{\infty} \frac{\beta_m}{3^m}$ . Following [5], we expand the sum as follows:

$$\begin{aligned} \sum_{m=0}^{\infty} \frac{\beta_m}{3^m} &= \beta \sum_{m=0}^{\infty} \frac{3^{-m}}{(3-1)(3^2-1)\cdots(3^m-1)} \\ &= \beta \sum_{m=0}^{\infty} \frac{3^{-m} 3^{-\frac{m(m+1)}{2}}}{(1-1/3)(1-(1/3)^2)\cdots(1-(1/3)^m)} \\ &= \beta \prod_{m=0}^{\infty} (1+(1/3)^{m+1}) \\ &= \prod_{m=0}^{\infty} \frac{(1+(1/3)^{m+1})}{(1+(1/3)^m)} \\ &= \frac{3}{4}. \end{aligned}$$

The third equality is just Lemma 4 of [5]. This gives the upper bound in Theorem 1.1.  $\square$

Additionally, here are the considerations that lead one to make Conjecture 1.2. The analogous prediction to Steinhagen's in [8] is that, for  $\alpha \in S_m(X)$ , there is a  $\frac{2}{3^{m+1}-1}$  chance that there is a solution to the  $\zeta_3$ -Pell equation for  $K(\sqrt[3]{\alpha})$ . Thus, one computes

$$\sum_{m=0}^{\infty} \frac{2\beta}{(3^{m+1}-1)\prod_{j=0}^m(3^j-1)} = 2 \sum_{m=1}^{\infty} \frac{\beta}{\prod_{j=0}^m(3^j-1)}$$

$$\begin{aligned}
&= 2 \left( \left( \sum_{m=0}^{\infty} \frac{\beta}{\prod_{j=0}^m (3^j - 1)} \right) - \beta \right) \\
&= 2 \left( \left( \sum_{m=0}^{\infty} \beta_m \right) - \beta \right) \\
&= 2(1 - \beta).
\end{aligned}$$

## 6. DIFFERENCES BETWEEN THE $\zeta_3$ -PELL EQUATION AND THE NEGATIVE PELL EQUATION

The goal of this section is two-fold. Firstly, we wish to convince the reader that naively generalizing Stevenhagen's conjecture may not be correct. Secondly, we wish to explain that things behave differently when we allow  $\lambda$  to ramify.

To the first end, we will discuss some ramifications of the considerations that Stevenhagen made in making his conjecture, and show that these don't port directly to the  $\zeta_3$ -Pell case. Consider fields  $L$  of the form  $\mathbb{Q}(\sqrt{pq})$  with  $p, q \equiv 1 \pmod{4}$ . Stevenhagen predicts that one half of the time,  $\text{Cl}(L)_2 = \mathbb{Z}/2\mathbb{Z}$  and there is a solution to the negative Pell equation. One half of the remaining time,  $\text{Cl}(L)_2 = \mathbb{Z}/2\mathbb{Z}$  but there is no solution to the negative Pell equation. One half of the remaining time,  $\text{Cl}(L)_2 = \mathbb{Z}/4\mathbb{Z}$  and there is a solution to the negative Pell equation, and so on.

Moving over to the case where  $L = K(\sqrt[3]{\pi_1\pi_2})$ , the story starts off the same: two-thirds of the time,  $\text{Cl}(L)_{\sigma-1} = R_{\sigma}/(\sigma-1)$  and there is a solution to the  $\zeta_3$ -Pell equation, and two-thirds of the remaining time  $\text{Cl}(L)_{\sigma-1} = R_{\sigma}/(\sigma-1)$  but there is no solution to the  $\zeta_3$ -Pell equation. However, we have the following fact that causes everything to go off the rails:

**Proposition 6.1.** *Assume that  $\left(\frac{\pi_1}{\pi_2}\right)_3 = 1$  and there is a solution to the  $\zeta_3$ -Pell equation in  $L$ . Then  $\text{Cl}(L)[(\sigma-1)^3] = R_{\sigma}/(\sigma-1)^3$ .*

This proposition implies that the case “ $\text{Cl}(L)_{\sigma-1} = R_{\sigma}/(\sigma-1)^2$ ,  $\zeta_3$ -Pell has a solution” is skipped. Numerical calculations seem to suggest that this is literally the only case that is skipped.

*Proof.* Since there is a solution to the  $\zeta_3$ -Pell equation, one has that  $\text{Cl}(L)[\sigma-1]$  is generated by  $\mathfrak{P}_1$ , the prime lying over  $\pi_1$ . Letting  $L_1 = K(\sqrt[3]{\pi_1}, \sqrt[3]{\pi_2})$  so that  $\text{Gal}(L_1/L) = \text{Cl}(L)/\sigma-1$ , one sees that the first R edei map is zero. Thus, there is a field  $L_2/L$  as in the middle of section 3 replacing 17 with  $\pi_1$  and  $\pi$  with  $\pi_2$ . Letting  $F = K(\sqrt[3]{\pi_1})$ ,  $\tau$  generate  $\text{Gal}(F/K)$ , and  $V = (\mathcal{O}_F/\pi_2)^{\times} \otimes_{\mathbb{Z}} \mathbb{F}_3$ , we have that  $\text{Gal}(L_2/F) = V_{\pi}/V_{\pi}[\tau-1]$ . Now, we need to compute the image of the Frobenius of the prime lying over  $\pi_1$  in  $\text{Gal}(L_2/L)$ . But this is zero if and only if that's the case for  $L_2/F$ . But then we know what that is: we can write  $V = ((\mathcal{O}_K/\pi_2)^{\times} \otimes_{\mathbb{Z}} \mathbb{F}_3)^3$  with  $\tau$  permuting the factors. Choosing an element  $a$  such that  $a^3 \equiv \pi_1 \pmod{(\pi_2)}$ , we get that the Frobenius is just the image of  $(a, \zeta_3 a, \zeta_3^2 a)$  in  $\text{Gal}(L_2/F)$ . By our long-running assumption on primes in  $K$ , we have that  $\zeta_3$  is a cube, so this lies in the diagonal of  $V$ . But  $V[\tau-1]$  is exactly equal to the diagonal, so the image of the Frobenius of the prime lying over  $\pi_1$  in  $\text{Gal}(L_2/L)$  is 0, showing the proposition.  $\square$

Now, we will discuss what can happen when we allow  $\lambda$  to ramify. Consider fields now of the form  $L = K(\sqrt[3]{\zeta_3\pi})$ . There is now a difference in what generates the  $\sigma-1$  torsion in the class group: since one has that the prime over  $\pi$  is just  $(\sqrt[3]{\zeta_3\pi})$ , we have that the natural candidate to check is only the prime over  $\lambda$ , which we will denote by  $\mathfrak{p}$ . One has that  $\mathfrak{p}$  splits in  $L(\sqrt[3]{\pi})$  if and only if  $\left(\frac{\lambda}{\pi}\right)_3 = 1$ , which by cubic reciprocity is equivalent to  $\pi \equiv 1$

(mod  $\lambda^4$ ). As above, we may ask if there is a field  $L_2/L$  unramified everywhere, which is the same as asking if there is a field  $L_2/K(\zeta_9)$  unramified outside of  $\pi$ . Letting  $F = K(\zeta_9)$  and  $\tau$  generate  $\text{Gal}(F/K)$ , the exact same considerations as before show that such a field exists if and only if  $\pi$  splits in  $F(\sqrt[3]{(\mathcal{O}_F^\times)^{\tau-1}})$ . But  $\mathcal{O}_F^\times$  is generated by  $-\zeta_9$ ,  $\frac{1-\zeta_9^4}{1-\zeta_9}$ , and  $\frac{1-\zeta_9^7}{1-\zeta_9}$ . An easy calculation shows that  $\zeta_9^{\tau-1} = \zeta_3$ ,  $\left(\frac{1-\zeta_9^4}{1-\zeta_9}\right)^{\tau-1} = \frac{\lambda}{(1-\zeta_9^4)^3}$ , and  $\left(\frac{1-\zeta_9^7}{1-\zeta_9}\right)^{\tau-1} = \frac{\lambda}{(1-\zeta_9^7)^3}$ , so the only conditions on having such an extension are that  $\zeta_3$  and  $\lambda$  are cubes, something that is already imposed. Thus, one skips now the case where “ $Cl(L)_{\sigma-1} = R_\sigma/(\sigma-1)$ ,  $\zeta_3$ -Pell has no solution.” Again, numerical calculations seem to suggest that this is the only case that is skipped.

What’s going on here seems to be a bit more straightforward, and is probably actually closer to typical. In the negative Pell case, one is not allowed to have ramification like  $\mathbb{Q}_2(i)/\mathbb{Q}_2$  at 2, because that provides a local obstruction to  $-1$  being a norm. But that is entirely specific to 2; for odd  $p$ ,  $N_{\mathbb{Q}(\zeta_{p^2})/\mathbb{Q}(\zeta_p)}(\zeta_{p^2}) = \zeta_p$  in contrast to  $N_{\mathbb{Q}(i)/\mathbb{Q}}(i) = 1$ . Other numerical calculations suggest that local ramification of the form  $K_\lambda(\sqrt[3]{\lambda})/K_\lambda$  at  $\lambda$  seems to fit into the earlier framework.

In summary, more calculations are needed, and there very well might need to be a pair of conjectures about the  $\zeta_3$ -Pell equation: one in the case where the ramification at  $\lambda$  is of the form  $K_\lambda(\zeta_9)/K_\lambda$  and one that covers every other case.

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