FiveThirtyEight's March 19, 2021 Riddler

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This week's riddler, courtesy of Scott Matlick, is about square numbers:

Question 1. 16 *is a square number, such that, when you remove the last digit of it, it remains square. The next few numbers that do this are* $49, 169, 256,$ *and* 361*. Can you find the next three such numbers?*

Extra Credit*:* 169 *has the property that when you remove either the last digit or the last two digits, you get a square number each time. Can you find another square with this property?*

Disclaimer: one interpretation of the question considers a zero-digit number to be 0 (assume that every number has infinitely many leading 0s), so that 1, 4, and 9 are also numbers with the property listed (remove the last digit and get a square). Additionally, one gets that 1, 4, 9, 16, and 49 all solve the extra credit. I will not ascribe to this interpretation throughout the rest of this solution.

Ironically, the extra credit is easier than the main question. Assume that x^2, y^2, and z^2 are the three squares that come up in the extra credit. Then one has that $0 < x^2 - 10y^2 < 10$ and $0 < y^2 - 10z^2 < 100$, and so $0 < x^2 - (10z)^2 < 100$. Since $x > 10z$, one has that $20z < x + 10z < x^2 - (10z)^2 < 100$ and so $z < 5$. One can therefore check all squares of the forms $1ab$, $4ab$, $9ab$, and $16ab$, and it becomes quickly apparent that 169 is the only such square.

On to the main question. Write x^2 and y^2 for the two squares. One has that $0 < x^2 - 10y^2 < 10$. It is easy to see that therea re at most five possibilities for $x^2 - 10y^2$: $1, 4, 5, 6,$ and 9. However, if the last digit of x^2 is 5, then the second to last digit is 2 and so y^2 is a square that ends in 2. Thus, one only needs to find all solutions to $x^2 - 10y^2 \in \{1, 4, 6, 9\}$. Using the arithmatic of $\mathbb{Z}[\sqrt{10}]$, one has that, if x and y are a solution to $x^2 - 10y^2 = 1$, then $x + y\sqrt{10} = (19 + 6\sqrt{10})^n$ for some $n > 0$. Similarly, one has that solutions to $x^2 - 10y^2 = 4$ come from $x + y\sqrt{10} = 2(19 + 6\sqrt{10})^n$ for some $n > 0$. Additionally, solutions to $x^2 - 10y^2 = 6$ come from $x + y\sqrt{10} = (4 + \sqrt{10})(19 + 6\sqrt{10})^n$ or $(16 + 5\sqrt{10})(19 + 6\sqrt{10})^n$ for some $n \geq 0$. Finally, solutions to $x^2 - 10y^2 = 9$ come from $x + y\sqrt{10} = (7 + 2\sqrt{10})(19 + 6\sqrt{10})^n$, $x + y\sqrt{10} = (13 + 4\sqrt{10})(19 + 6\sqrt{10})^n$, or $x + y\sqrt{10} = 3(19 + 6\sqrt{10})^n$, where in the first two instances, one has $n \geq 0$ and in the last one one has $n > 0$.

One gets that the next three examples are $38^2 - 10 \cdot 12^2 = 4$, $57^2 - 10 \cdot 18^2 = 9$, and $136^2 - 10 \cdot 43^2 = 6$. The two after that are $253^2 - 10 \cdot 80^2 = 9$ and $487^2 - 10 \cdot 154^2 = 9$. You can generate all of the solutions with this process.