FiveThirtyEight's Janurary 7, 2022 Riddler

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This week's riddler is a geometry challenge:

Question 1. *Amare the ant is traveling within Triangle ABC. Angle A measures 15 degrees, and sides AB and AC both have length 1.*

Amare starts at point B and wants to ultimately arrive on side AC. However, the queen of his colony has asked him to make several stops along the way. Specifically, his path must:

- *Start at point B.*
- *Second, touch a point on side AC*
- *Third, touch a point back on side AB.*
- *Finally, touch a point on AC.*

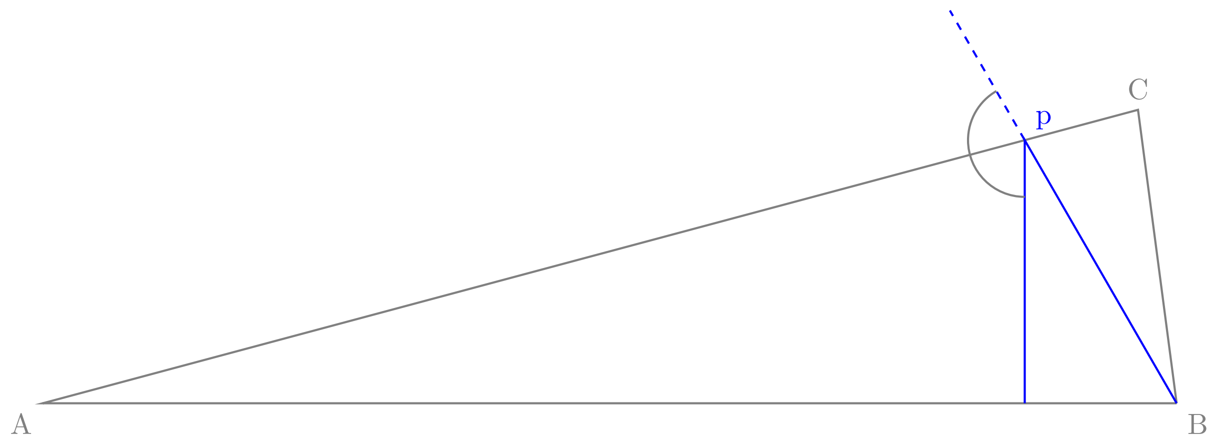
What is the shortest distance Amare can travel to appease the bizzare queen?

There is a very short answer that I will present shortly. However, I did five 5 pages of calculus before coming up with that solution so I feel obliged to share this with you.

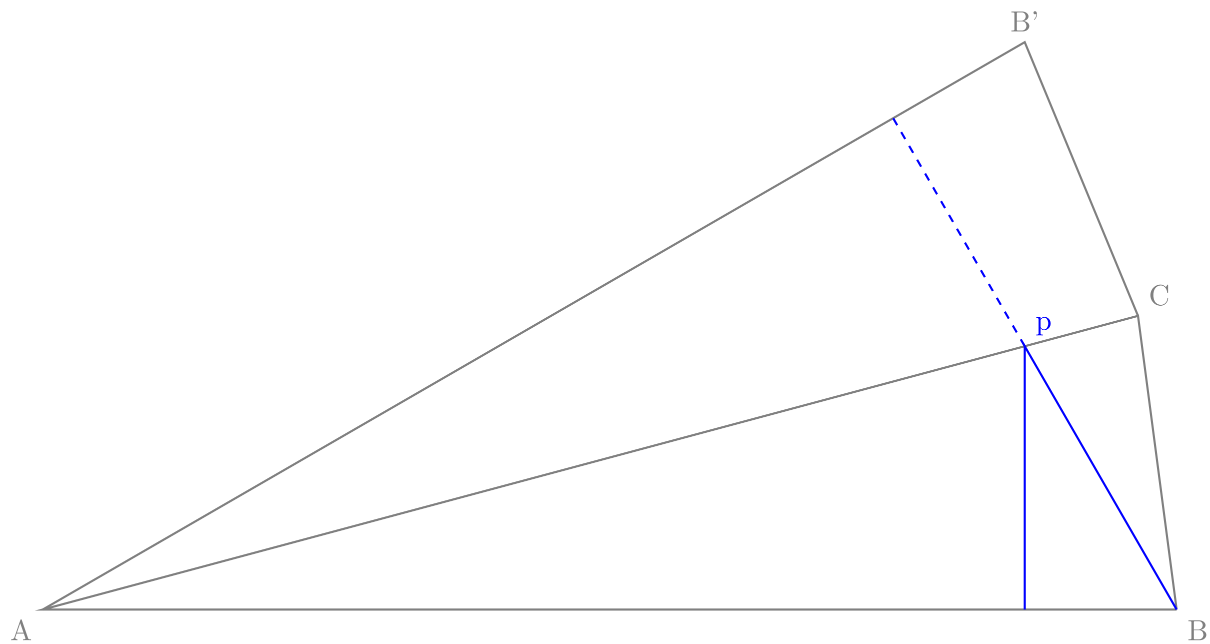
I will solve the problem where Amare goes from B to AC to AB and stops there, as that is a building block to getting the three stop problem. It is clear that once Amare touches side AC, his best path is to travel straight down to the edge AB. Let p be a point on the edge AC (actually, it's better to think of this as a point in $\mathbb{R}^2$ for reasons that will become apparent shortly), and define d_1 to be the distance to B, and d_2 to be the distance to AB. I will view d_1 and d_2 as functions of p. One has that ∇d_1 is the unit vector in the direction from B to p, and $\nabla d_2 = (0, 1)$, the unit vector pointing straight up.

To figure out where on AC Amare should go, let v be a vector parellel to AC. At the correct point, one has that $v \cdot (\nabla d_1 + \nabla d_2) = 0$ (as that is just setting the directional derivative to 0). Since ∇d_1 and ∇d_2 are both unit vectors, the only relevant thing is that the angle between Bp and v is equal to the angle between v and $(0, -1)$; a quick calculation shows that this gives the angle between AB and Bp is $60°$.

To see a picture, the optimal path is as follows:

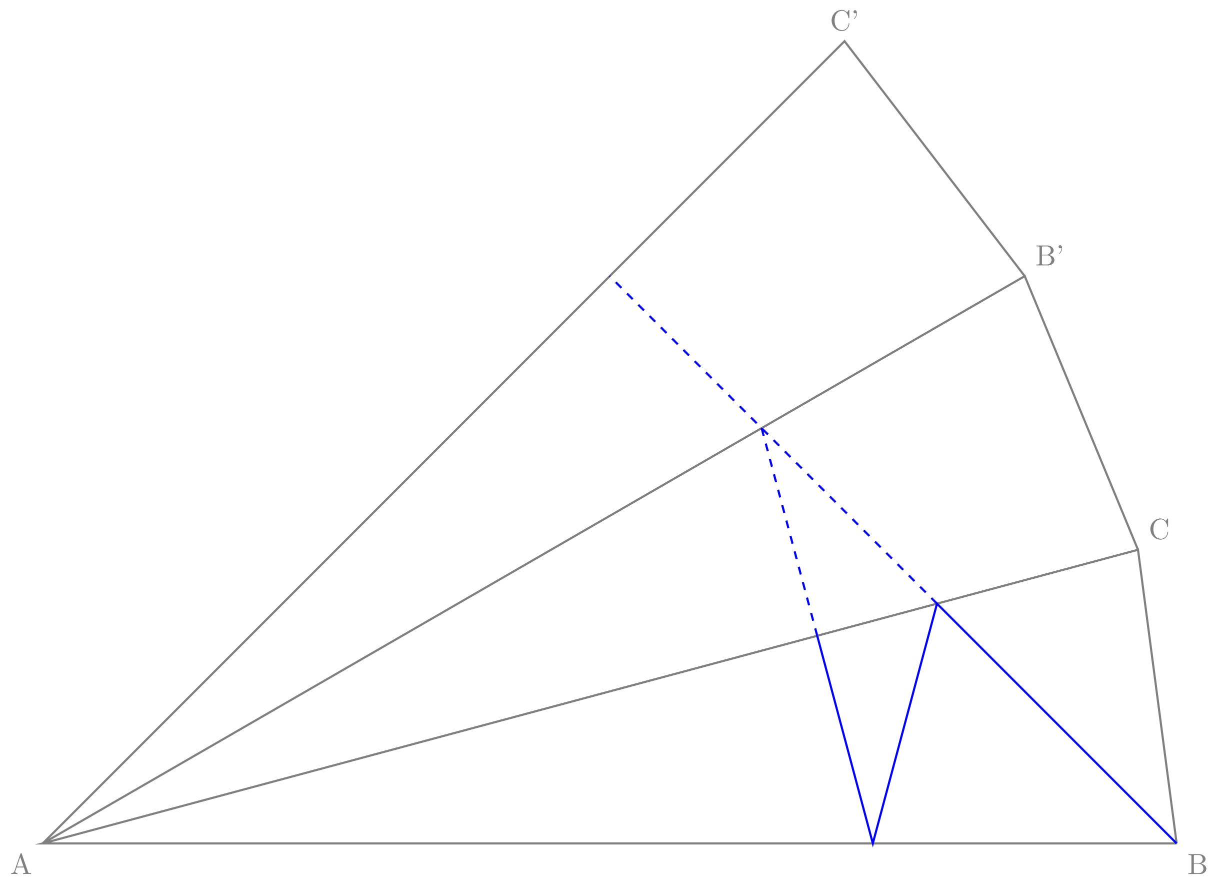


This image leads one to imagine AC as a mirror. Drawing a reflected triangle across AC, one instantly sees what's really going on:



The fastest path for Amare is to just go directly to AB'.

For the three-step version (i.e. what was actually asked for), it's no surprise that you just draw another triangle reflected about AB', and go directly to AC':



With this picture, it's immediately apparent that Amare moves $\frac{\sqrt{2}}{2}$ units, as the point of intersection on AC' is $(\frac{1}{2}, \frac{1}{2})$.