FiveThirtyEight's April 15, 2022 Riddler

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This week's riddler, courtesy of Daniel Larsen, is about Carmichael numbers:

Question 1. *Can you find a Carmichael number which has six digits in base* 10*, and is of the form* $ABCABC$ *with* A*,* B*, and*C *being digits?*

Write $x = ABC$ in base 10. Then, the question is "Can you find an integer x such that $100 \leq x \leq 999$ and $1001x$ is a Carmichael nummber?"

Since $1001 = 7 \cdot 11 \cdot 13$, one needs that $1001x$ must be congruent to $1 \pmod{60}$, and so $x \equiv 41 \pmod{60}$. There are then fifteen numbers to check. While it is possible to do this by hand, it's easier to break it into two cases depending on whether x is prime or composite.

If x is prime, then there is one more condition for $1001x$ to be a Carmichael number: $1001x \equiv 1 \pmod{x-1}$. This is equivalent to $1001 \equiv 1 \pmod{x-1}$, or $x - 1|1000$. There is only one such choice of x that is also $41 \pmod{60}$: $x = 101$. This gives 101101 as a solution to the problem.

If x is composite, let p be the smallest prime divisor of x (which can be at most 29). By construction, $p \neq 2, 3$, or 5. Additionally, $p \neq 7, 11$, or 13 because Carmichael numbers are squarefree. Additionally, if $p = 23$ (or 29), then $1001x - 1$ must be divisible by 11 (or 7) which is impossible, as $11|1001$ (or $7|1001$). Thus, $p = 17$ or $p = 19$.

If $p = 17$, then the smallest integer congruent to $41 \pmod{60}$ is 221. But that doesn't work, as $221 = 13 \cdot 17$. The second smallest such number is 1411, which is too large. Thus, $p \neq 17$. If $p = 19$, then the smallest choice of x is 1121 which is too large.

Thus, the only such number is 101101.