FiveThirtyEight's July 31, 2020 Riddler

Emma Knight

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This weeks riddler, courtesy of Dave Moran, is about functions on $\mathbb{Z}/n\mathbb{Z}$:

Question 1. *Find the smallest $n > 100$ such that there doesn't exist a function $f : \mathbb{Z}/n\mathbb{Z}$ such that*

- *f is a permutation,*
- *$f(x) = x$ implies $x = 0$, and*
- *$f(x_1) - f(x_2) \neq x_1 - x_2$ for all $x_1 \neq x_2 \in \mathbb{Z}/n\mathbb{Z}$.*

The motivation is about a class standing in a circle in a maximally disordered state.

Let's assume that f is a function that satisfies the conditions in the question. Notice first that, if $g : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ is defined by $g(x) = f(x) - x$, then g is also a permutation ($g(x_1) = g(x_2)$ can be easily rearanged to $f(x_1) - f(x_2) = x_1 - x_2$). Also, notice that $\sum_{x \in \mathbb{Z}/n\mathbb{Z}} f(x) = \sum_{x \in \mathbb{Z}/n\mathbb{Z}} x = \frac{n(n+1)}{2}$ (the first equality is because f is just rearranging the elements of $\mathbb{Z}/n\mathbb{Z}$), and the same is true for g because g is also a permutation. But one also has that $\sum_{x \in \mathbb{Z}/n\mathbb{Z}} g(x) = \sum_{x \in \mathbb{Z}/n\mathbb{Z}} f(x) - \sum_{x \in \mathbb{Z}/n\mathbb{Z}} x = 0$. Thus, one must have that $\frac{n(n+1)}{2} = 0$ in $\mathbb{Z}/n\mathbb{Z}$. Thus, if such a function exists, then n cannot be even, as $\frac{n(n+1)}{2} = \frac{n}{2} \neq 0$ in $\mathbb{Z}/n\mathbb{Z}$.

Now, one can easily see that, if n is odd, then $f(x) = 2x$ satisfies all of the criteria for the question. Thus, one gets that the set of all n such that there does not exist such a function is exactly the set of all even n. In particular, the minimum value of n greater than 100 is 102.