FiveThirtyEight's February 19, 2021 Riddler

Emma Knight

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This week's riddler is about baking pie:

Question 1. *You are baking a pie, and you have a fixed amount of dough to use for crust. If your pie is a cylinder, what fraction of the crust should you use on the bottom of the pie to maximize the volume of the pie?*

Assume that the pie has radius r and height h. Then the total amount of crust you have is $A = 2\pi r^2 + 2\pi rh$ and the total volume is $V = \pi r^2 h$.

Solution 1: This solution uses only material taught in calc I. Looking along the curve $A = c$, one can compute $0 = \frac{dA}{dr} = 4\pi r + 2\pi h + 2\pi r\frac{dh}{dr}$, and gets that $\frac{dh}{dr} = -2 - \frac{h}{r}$. This means that $\frac{dV}{dr} = 2\pi rh + \pi r^2 \left(\frac{dh}{dr}\right) = 2\pi rh - 2\pi r^2 - \pi rh = \pi r(h - 2r)$. Setting $\frac{dV}{dr} = 0$, one gets that $r = 0$ or $h = 2r$. Clearly the first solution is a minimum, and the relevant solution is $h = 2r$, so $A = 6\pi r^2$. Since the bottom of the pie has area πr^2, you should set aside one-sixth of the crust for the bottom.

Solution 2: This is a multivariable calc solution. Write x for the area of the bottom, and y for the area of the side. One has that $A = 2x + y$ and $4\pi V^2 = xy^2$; since maximizing V and maximizing $4\pi V^2$ are the same thing, I will do the latter.

We need to maximize xy^2 subject to $2x + y = c$; using Lagrange multipliers, one gets the equations $y^2 = 2\lambda$ and $2xy = \lambda$. Equating these two equations, we get $y^2 = 4xy$ so either $y = 0$ (clearly not a maximum) or $y = 4x$. Thus, the area is $6x$, so, as before, we should set aside one-sixth of the crust for the bottom.

All told, this pie looks more like an enclosed canoli than an actual pie.