FiveThirtyEight's March 26, 2021 Riddler

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This week's riddler is about a variant of basketball:

Question 1. *The rules for men's basketball in the Riddler Collegiate Athletic Association's (RCAA) are a little different from those in the NCAA. In the NCAA, when a player is fouled attempting a 3-point shot and misses, they always get three free throw attempts, regardless of how many fouls the opposing team has committed.*

But in the RCAA, a player must earn each additional foul shot by making the previous one. In other words, a player can take a second shot if they make the first, and they can take a third shot if they make the second.

Suppose a player on your team has a known shooting profile: Their probability of making the first free throw is p, their probability of making the second is q, and their probability of making the third is r, such that no two of these probabilities are equal. Meanwhile, their expected number of points made for any given three-point foul (which can be computed from p, q and r) is also known.

What is the greatest number of distinct shooting profiles that are made up of these three different probabilities - p, q and r, in some order for the three shots — that can result in the same overall expected number of points?

Let $f(x, y, z)$ be the expected number of points scored assuming that the probabilities are x, y, and z for the first, second, and third shot respectively. Then one has that $f(x, y, z) = x + xy + xyz$, as the probability that you score the first point is x, the probability that you score the second point is xy (you have an x chance to take it and a y chance to make it) and the probability that you score the third point is xyz. Now, since the last term there is the same if you permute x, y, and z, I will also define $g(x, y, z) = x + xy$.

Now let p, q, and r be fixed distinct probabilities (i.e. real numbers in $[0, 1]$). We want to see how many possible ways to put p, q, and r into g give the same value. Trivially, if $p = 0$, then $g(p, q, r) = g(p, r, q) = 0$, so two is always possible[1]. Now, I want to show the following:

1. $g(p, q, r) = g(q, p, r)$ is impossible.

[1] This isn't the only way to get two; this is just the easiest. One can also see that $p = .3$, $q = .2$, and $r = .8$ gives $g(p, q, r) = g(q, r, p) = .36$.

2. $g(p, q, r) = g(r, q, p)$ is impossible.

3. If $g(p, q, r) = g(p, r, q)$, then $p = 0$ and no other permutation of p, q, and r evaluate to 0 in g.

If $g(p, q, r) = g(q, p, r)$, then $p + pq = q + pq$ so $p = q$, a contradiction. Similarly, if $g(p, q, r) = g(r, q, p)$, then $p(1 + q) = r(1 + q)$. Since $q \geq 0$, one has that $p = r$, again impossible. If $g(p, q, r) = g(p, r, q)$, then $p + pq = p + pr$, so $p = 0$ or $q = r$. Since the second is impossible, one has that $p = 0$. Now, for any other permutation, since $g(x, y, z) = x(1 + y)$, we are plugging a positive number in for x and a non-negative number in for y, we get that $g(x, y, z) \neq 0$ for any other permutation.

Thus, if you have at least three possibilities, then you can't have a swap among the permutations, so you must have $g(p, q, r) = g(q, r, p) = g(r, p, q)$ (after renaming). Thus, we have

$$p + pq = q + qr = r + rp.$$

Now, one gets

$$\begin{aligned} qr(1 + p) &= q(p + pq) \\ &= pq + pq^2 \end{aligned}$$

$$\begin{aligned} qr(1 + p) &= (p + pq - q)(1 + p) \\ &= p + p^2 + pq + p^2q - q - pq \\ &= p - q + p^2 + p^2q \end{aligned}$$

$$\begin{aligned} pq + pq^2 &= p - q + p^2 + p^2q \\ 0 &= p - q + p^2 - pq + p^2q - pq^2 \\ &= (p - q)(1 + p + pq). \end{aligned}$$

Since $p \neq q$, one has that $1 + p + pq = 0$, or $q = -\frac{1+p}{p}$. But p must necessarily be positive, so that gives q being negative, which is also impossible. Thus, there is no way to have three possible shooting profiles have the same expected number of points.