FiveThirtyEight's March 25, 2022 Riddler

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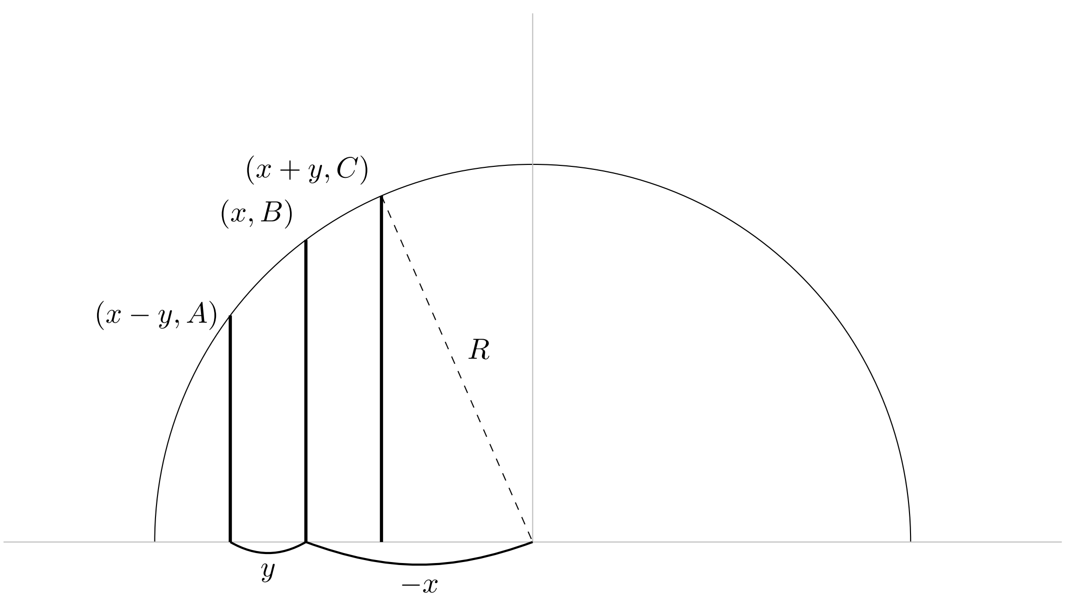
This week's riddler is an astronomy puzzle:

Question 1. *The astronomers of Planet Xiddler are back at it. They have developed a new technology for measuring the radius of a planet by analyzing its cross sections.*

And so, they launch a satellite to study a newly discovered, spherical planet. The satellite sends back data about three parallel, equally spaced circular cross sections which have radii A, B and C megameters, with $0 < A < B < C$. Based on these values, the scientists calculate the radius of the planet is R megameters. To their astonishment, they find that A, B, C and R are all whole numbers!

What is the smallest possible radius of the newly discovered planet?

The three-dimensionality of this problem isn't actually needed; this can be solved by looking at a cross-section of the planet that is perpendicular to the original three cross sections, and passing through the origin. Then one has the following picture:



One then has the following equations:

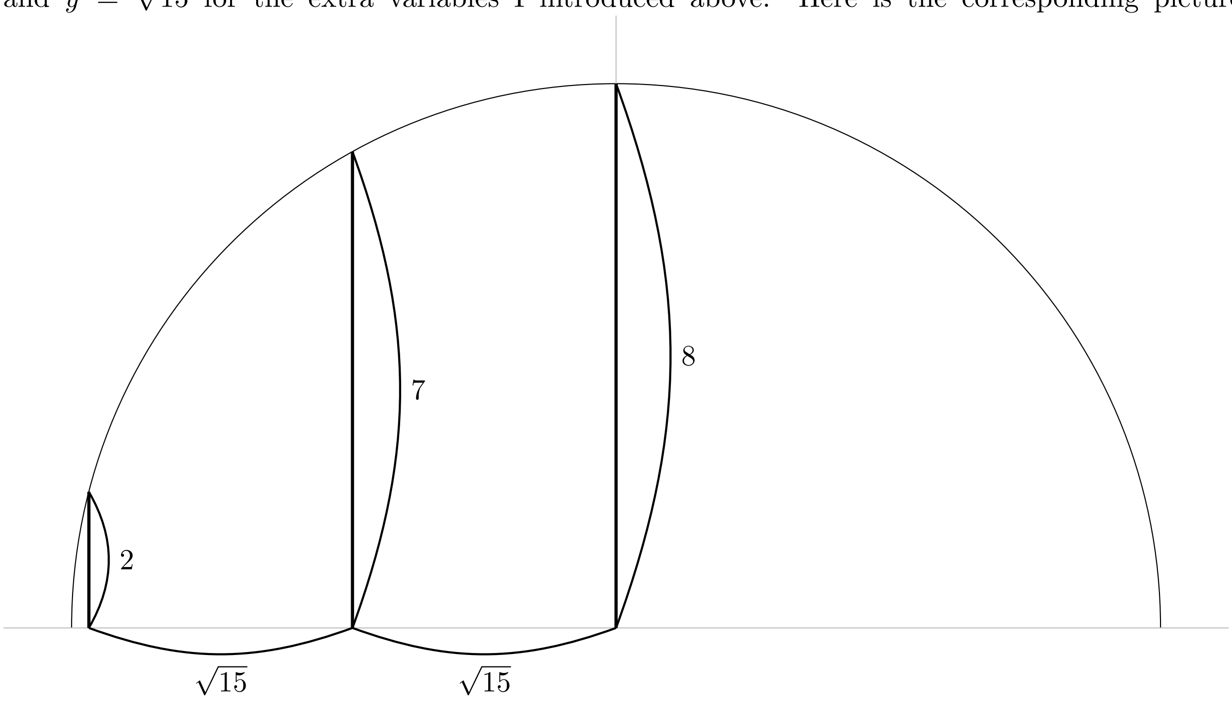
$$
\begin{aligned}
A^2 + (x-y)^2 &= R^2 \\
B^2 + x^2 &= R^2 \\
C^2 + (x+y)^2 &= R^2.
\end{aligned}
$$

One gets in order:

$$
\begin{aligned}
A^2 + C^2 + 2x^2 + 2y^2 &= 2B^2 + 2x^2 \\
y^2 &= B^2 - \frac{A^2 + C^2}{2} \\
C^2 + x^2 + 2xy + y^2 &= B^2 + x^2 \\
2xy &= \frac{A^2 - C^2}{2} \\
x &= \frac{A^2 - C^2}{4y} \\
R^2 &= B^2 + \frac{(A^2 - C^2)^2}{16B^2 - 8A^2 - 8C^2}.
\end{aligned}
$$

In order for R to be an integer, one needs that quantity on the right to be a perfect square of an integer. Additionally, for this to be a real circle one needs $y^2 > 0$. At this point, it is not too hard to loop over values of A, B, and C to find integer values. Additionally, since $C < R$, once you find one such quadruple, you only need to continue searching until you exhaust all values with C less than that value of R.

It turns out that the smallest such quadruple is $(2, 7, 8, 8)$ for (A, B, C, R); this gives $x = -\sqrt{15}$ and $y = \sqrt{15}$ for the extra variables I introduced above. Here is the corresponding picture:



Finally, here is the code:

```
import itertools
import math

##top is the highest value of C I search for.  solns
##is a list of all the valid triples (A, B, C, R).
##small is the smallest value of R I have found so far.
##best is the triple with the smallest value of R.
##checks is a list of all possible triples (A, B, C)

top = 100
solns = []
small = top*2
best = []
checks = itertools.combinations(range(1, top+1), 3)

##This loop finds all solutions and determines which one
##has the smallest value of R.

for check in checks:
    a = check[0]
    b = check[1]
    c = check[2]
    if(16*(b**2)-8*(a**2)-8*(c**2) > 0):
        if(((a**2 - c**2)**2) % (16*(b**2)-8*(a**2)-8*(c**2)) == 0):
            r = math.sqrt(((a**2 - c**2)**2) // (16*(b**2)-8*(a**2)-8*(c**2)) + b**2)
            if r.is_integer():
                solns.append([a, b, c, r])
                if (r < small):
                    small = r
                    best = [a, b, c, r]

print(solns)
print(best)
```