FiveThirtyEight's January 17, 2020 Riddler

Emma Knight

January 23, 2020

This is a complete solution to the riddler from January 17, 2020.

Question 1. *Two ducks are sitting on the middle rock of a 3 by 3 grid of rocks. Every minute, each duck randomly swims to an adjacent rock with equal probability (so if a duck is in the center, then it swims to one of the four edge rocks with a probability of $1/4$ for each, if it's on an edge then it swims to the center with probability $1/3$ and one of the two adjacent corners with probability $1/3$ each, and if it's on a corner it swims to one of the two adjacent edges with probability $1/2$). How much time on average does it take for the two ducks to rendez-vous again?*

The first observation is that a duck can only move from the center or a corner to an edge, and can only move from an edge to the center or a corner. So while one might think that there are many different possible configuartions of the ducks, this number is cut in half. Additionally, the symmetries of a square also cuts down how many states are possible, and the full list is easily seen by just enumerating them out:

Claim 2. *There are 5 different states for the ducks to be in. They are:*

- *Two opposite corners,*
- *Two adjacent corners,*
- *One corner and the center,*
- *Two opposite edges, and*
- *Two adjacent edges.*

I will call these states OC, AC, CC, OE, and AE respectively. This is a Markov chain, with the following transition matrix:

$$M = \begin{pmatrix} & & & \frac{1}{2} & \frac{1}{2} \\ & & & \frac{1}{4} & \frac{1}{2} \\ & & & \frac{1}{4} & \frac{1}{2} \\ \frac{2}{9} & \frac{2}{9} & \frac{4}{9} & & \\ \frac{1}{9} & \frac{2}{9} & \frac{4}{9} & & \end{pmatrix}$$

For the readers unfamiliar with Markov chains, the way that this matrix is to be read is as follows: the probability of moving from OC (the first state) to OE (the fourth state) is found by looking at the coefficient in the first row and the fourth column. Any coefficients that are blank are just 0. Finally, I'm suppressing the odds that two ducks move onto the same square.

Now, write t_{OC} to be the expected amount of minutes it takes for the two ducks to move onto the same square given that at minute 0 they are on opposite corners (and similar variables for the other states). It's not too hard to see that one gets the following system of equations:

$$\begin{pmatrix} t_{OC} \\ t_{AC} \\ t_{CC} \\ t_{OE} \\ t_{AE} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} & & & \frac{1}{2} & \frac{1}{2} \\ & & & \frac{1}{4} & \frac{1}{2} \\ & & & \frac{1}{4} & \frac{1}{2} \\ \frac{2}{9} & \frac{2}{9} & \frac{4}{9} & & \\ \frac{1}{9} & \frac{2}{9} & \frac{4}{9} & & \end{pmatrix} \begin{pmatrix} t_{OC} \\ t_{AC} \\ t_{CC} \\ t_{OE} \\ t_{AE} \end{pmatrix}$$

Some simple manipulations turns this into $(I - M)v = v_0$, with v being the vector with the five unknowns (t_{OC} et. al.) and v_0 being the vector with all 1s. Thus, $v = (I - M)^{-1}v_0$, and since these are specific matrices, it's an easy exercise to ask a computer to compute this. The answer is

$$v = \begin{pmatrix} \frac{234}{37} \\ \frac{363}{74} \\ \frac{363}{74} \\ \frac{210}{37} \\ \frac{184}{37} \end{pmatrix}.$$

Finally, the answer that we are looking for is $1 + \frac{1}{2}t_{AE} + \frac{1}{4}t_{OE}$: after one minute, there is a $\frac{1}{4}$ chance that the ducks meet again, a $\frac{1}{2}$ chance that they are in adjacent edges, and a $\frac{1}{4}$ chance that they are in opposite edges. This number is just $\frac{363}{74}$. This works out to being about 4 minutes and 54 seconds.

One can also see what the amount of time is for other starting positions: if both ducks start on an edge, then the expected amount of time is $1 + \frac{2}{9}t_{AC} + \frac{4}{9}t_{CC} = \frac{158}{37}$ or about 4 minutes and 16 seconds. If both ducks start on a corner, then the expected amount of time is $1 + \frac{1}{2}t_{AE} = \frac{129}{37}$, or about 3 minutes and 29 seconds.